Development and Evaluation of Solution Procedures for Geometrically Nonlinear Structural Analysis

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This paper presents a comparative study of the solution techniques which are applicable to the solution of the nonlinear algebraic or differential equations characterizing geometrically nonlinear structural behavior. After reviewing the currently available solution techniques, attention is focused at developing procedures which are computationally economical. The new solution procedures are compared numerically with some of the more conventional procedures such as the Newton-Raphson method, incremental methods, and iterational procedures. Application is made to highly nonlinear problems including a simple truss-spring problem and symmetrically and asymmetrically loaded shells of revolution. Results indicate that a new self-correcting initial-value formulation has possibilities as a powerful tool for nonlinear analysis.

Nomenclature

C = scalar amplifying factor in self-correcting initial-value solution
$[K]$ = linear stiffness matrix
$[K^*(q)]$ = nonlinear stiffness matrix which is a functon of q
P = scalar load parameter
$\{Q\}$ = generalized forces due to applied load
$\{\bar{Q}\}$ = normalized or scaled generalized force vector defined by $\{Q\} = \bar{P}\{\bar{Q}\}$
$\{Q^*(q)\}$ = pseudogeneralized forces due to nonlinearities in analysis
$\{Q_U\}$ = unbalance in equilibrium forces
$\{q\}$ = generalized displacements
U_L = strain energy based on linear strain-displacement relations
U_{NL} = strain energy contribution due to the inclusion of nonlinearities in the strain-displacement relations
Z = scalar amplifying factor in self-correcting initial-value solution
Δ = increment of quantity following symbol
$\{\sigma\}$ = stress vector
$\{\varepsilon\}$ = nonlinear strain vector
$(\dot{\ })$ = derivative of quantity in parentheses with respect to $\bar{P}$

Introduction

THE analysis of geometrically nonlinear structural problems has been a subject of considerable interest for over a decade, starting with the pioneering paper of Turner, Dill, Martin, and Melosh.[1] Essentially, the solution of a nonlinear problem reduces to that of tracing a nonlinear load displacement path by solving a system of nonlinear algebraic or differential equations. Accordingly, many different solution schemes for solving the governing equations have undergone development. The purpose of this paper is: 1) to summarize the solution procedures developed to date; 2) to propose some alternate solution methods; and 3) to compare and evaluate these procedures for a number of highly nonlinear structural problems. Attention is restricted primarily to procedures embodying the finite element stiffness method and only geometric nonlinearities are considered.

Until about 1968, researchers had little choice in selecting a solution procedure to solve the geometrically nonlinear problem. Until that time, only three solution procedures had been utilized—the incremental stiffness procedure,[1-4] iteration or successive substitution,[5-7] and the Newton-Raphson method.[8,9] The development and use of these methods was somewhat a direct extension from linear analyses and they evolved in a natural manner from the very simple step-by-step linear incremental technique. The past three to four years have seen, however, a great abundance of procedures for attacking the nonlinear equilibrium equations. These include iteration combined with systematic relaxation,[6,10,11] perturbation methods,[12,13] modified Newton-Raphson procedures,[4,14-16] self-correcting incremental forms,[16,17] incremental procedures combined with Newton-Raphson iteration,[18-21] initial value formulations,[4,16,22-25] and self-correcting initial-value formulations.[26,27] With this variety of solution procedures naturally comes the question of which solution technique is best suited for a particular application. The answer to this question of course hinges upon many factors: the type of problem to be solved, the degree of nonlinearity involved, the accuracy desired, the familiarity of the investigator with nonlinear analysis, the ease of application to automatic computation, the computer time required for a solution, and so on. This paper will attempt to answer this question through comparison of the various formulations in equation form and through comparative evaluation of the procedures when applied to highly nonlinear problems. The advantages of particular methods will be discussed.

Historical Background

Many authors have applied the finite element and finite difference methods to geometrically nonlinear problems as evidenced by recent survey articles.[4,28,29] From this work it is convenient to separate the solution procedures into two classes: Class I—methods which are incremental in nature and do not necessarily satisfy equilibrium; and class II—methods which are self-correcting and tend to stay on the true equilibrium path.

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Historically, class I was the first finite element approach to solving geometrically nonlinear problems.[1,2] In this method the load is applied as a sequence of sufficiently small increments so that the structure can be assumed to respond linearly during each increment. For each increment of load, increments of displacements and corresponding increments of stress and strain are computed. These incremental quantities are used to compute various corrective stiffness matrices (variously termed geometric, initial stress, and initial strain matrices) which serve to take into account the deformed geometry of the structure. A subsequent increment of load is applied and the process continued until the desired number of load increments has been applied. The net effect is to solve a sequence of linear problems wherein the stiffness properties are recomputed based on the current geometry prior to each load increment. The solution procedure takes the following form mathematically

$$[K+K_I]_{i-1}\{\Delta q\}_i = \{\Delta Q\} \tag{1}$$

where $[K]$ is the linear stiffness matrix, $[K_I]$ is an incremental stiffness matrix based upon displacements at load step $i-1$, $\{\Delta q\}$ is the increment of displacement due to the ith load increment, and $\{\Delta Q\}$ is the increment of load applied. The correct form of the incremental stiffness matrix has been a point of some controversy. Marcal[3] separates $[K_I]$ into initial stress and displacement matrices but neglects quadratic terms of the initial displacements. Reference 9 on the other hand includes these higher order terms. Murray and Wilson[21] present yet another form of the incremental approach.

The incremental approach is no doubt quite popular. This is due to the ease with which the procedure may be applied. However, the procedure has a serious disadvantage in that no real estimate of the solution accuracy is known since, in general, equilibrium is not satisfied at any given load level. This is evidenced by the drifting of the solution from the true solution. Recourse must be made to solving repeatedly the same problem with successively smaller load increments until convergence of two successive solutions can be established. In addition, for structures requiring many degrees of freedom, the updating of the incremental stiffness matrix plus the inversion of the new coefficient matrix at each load step may become excessively time consuming.

In another class I type procedure, Thompson and Walker[30] have applied the perturbation method to nonlinear problems. In this procedure, the incremental displacements are expanded in a Taylor series with respect to some incremental load parameter and about some known or assumed equilibrium state. Equations are obtained in the form

$$\{q\}_{i+1} = \{q\}_i + \{\dot{\Delta q}\}_i \Delta\bar{P} + \tfrac{1}{2}\{\ddot{\Delta q}\}_i \Delta\bar{P}^2 + \ldots \tag{2}$$

where the dot denotes the derivative with respect to the load parameter, $\bar{P}$; $\{\dot{\Delta q}\}$, $\{\ddot{\Delta q}\}$, etc. are path derivatives; and i denotes the load increment index. The terms in the Taylor series are obtained through the solution of several sets of linear equations equal in number to the number of terms retained in the expansion. Once the displacements are obtained at a particular load value, the whole process is repeated to obtain the displacements at the next load value. The procedure may of course drift from the true solution since errors will tend to accumulate. The amount of drift is dependent upon the load step size and the number of terms retained in the expansion. The procedure is straightforward but may become time consuming because of the numerous evaluations of the path derivatives. This method is also limited to problems wherein nonlinearities are not too large. Further development of this method is described in Refs. 12 and 13.

A final class I type procedure is the initial-value formulation.[16,22–25] This approach again treats the displacements and loads as a function of some load parameter, $\bar{P}$, such that $\{Q\} = \bar{P}\{\bar{Q}\}$. By differentiating the equilibrium equations with respect to $\bar{P}$, a set of differential equations is obtained in the form

$$[\bar{K}]\{dq/d\bar{P}\} = \{\bar{Q}\} \tag{3}$$

where $[\bar{K}]$ is a nonlinear stiffness matrix dependent upon displacements, $\{q\}$, and $\{\bar{Q}\}$ is a vector of scaled or normalized generalized forces. Values of $\{q\}$ at any load $\bar{P}$ can be obtained by numerical integration from a known initial displacement state. If the simple Euler method is used for the integration, then the incremental approach given by Eq. (1) is obtained. More accurate integration schemes such as the Runge-Kutta method or the predictor-corrector method may be used to reduce the drifting effect which is so prominent with the Euler integration.

The remaining solution techniques appear to be of the class II type. These procedures all make use of some method by which equilibrium is assured at any given point on the load displacement curve. These procedures are perhaps best described as self-correcting.

The iterational approach to solving the governing nonlinear algebraic equations has been used by many investigators.[5–7] This approach is relatively simply to apply. Starting with an initial estimate to the displacement solution, the nonlinear effects are estimated and a set of linearized equations is solved to obtain an improved solution. This solution is back-substituted into the equations and the iteration continued until convergence of successive iterations is obtained. The success of the method depends to a large extent upon the accuracy of the initial estimate of the displacements. The load may be applied in increments and various extrapolation procedures utilized to obtain accurate estimates. Relaxation schemes[6,10,11] may be used to accelerate convergence. While the iterational method is extremely fast from a computational standpoint, it has a serious disadvantage in that it will converge only for moderately nonlinear problems.[16]

In order to obtain convergence for problems exhibiting high nonlinearity, many investigators have utilized the Newton-Raphson iterational approach. This procedure is extremely accurate and usually converges quite rapidly for realistic initial estimates of the solution. Its chief drawback is the excessive computational effort required to form the coefficient matrix and invert it at each iterational cycle. Most investigators,[4,14–16] now use a modified Newton-Raphson procedure wherein the coefficient matrix is held constant for a number of iterations and then updated after the convergence rate has begun to deteriorate. Again, various extrapolation and relaxation procedures can be incorporated into the iterational cycle to insure and accelerate convergence.[31]

Recently, self-correcting forms of the incremental stiffness procedure have been developed by a number of investigators. References 19 and 20 describe combined procedures wherein the incremental stiffness procedure is used for a certain number of load steps and then equilibrium is corrected for by applying Newton-Raphson iteration. Murray and Wilson[21] determine the unbalance in nodal forces at the end of a load increment and then use an iterational approach to reduce the unbalance to zero. Their procedure is essentially a modified Newton-Raphson approach. References 16 and 17 formulate the incremental equations so that the out-of-balance in the equilibrium forces is explicitly taken into account. The resulting self-correcting incremental procedure has the advantage in that it is as easy to apply as the standard incremental procedure but is much more accurate.

The newest development in solution procedures appears to be a self-correcting initial-value formulation proposed by Stricklin, Haisler, and Von Riesemann[26] and Massett and Stricklin.[27] The procedure is applicable to highly nonlinear problems, is computationally economical, and relatively accurate. This procedure will be considered in some detail in the section on solution procedures.

Formulation

This section will present the development of the equilibrium equations through an application of the principle of virtual work and the solution procedures which may be utilized to solve these equations.

Equilibrium Equations

Many investigators choose to write the equilibrium equations in terms of the deformed coordinates of the body and, in doing so, they obtain expressions which depend upon the deformed geometry of the body. However, in the present formulation, the equilibrium equations will be written in terms of the original, undeformed coordinates. Such a formulation has an advantage in that all quantities are referred to the original body.

Without presenting the details, it is stated here that the principle of virtual work may be written so that it is valid for large strains but written in terms of the undeformed configuration.[32] This is easily seen by considering the following. The equilibrium equations are first written for the deformed body and then transformed so that differentiation is with respect to the undeformed Cartesian coordinates (see Novozhilov,[33] for example). The equilibrium equations in each of the coordinate directions are then multiplied by corresponding virtual displacements and the sum of the resulting equations integrated over the undeformed body. After some rearranging and an application of the divergence theorem, one obtains the virtual work equation as

$$\int_V \{\sigma^*\}^T\{\delta\varepsilon\}\, dV = \int_{S^*} \{p^*\}^T\{\delta u\}\, dS^* \tag{4}$$

where $\{\sigma^*\}$ are pseudostresses referred to an element of volume of the body before deformation, $\{\varepsilon\}$ are the complete nonlinear strain-displacement equations, $\{p^*\}$ are boundary surface forces acting in the undeformed coordinate directions and on the element of deformed surface, dS^*, $\{u\}$ are displacement components in the undeformed coordinate directions, and V is the undeformed volume. Body forces have been neglected in Eq. (4). Equation (4) is valid for large strains. The left and right side of Eq. (4) are normally defined as the variation of the strain energy, δU, and the work potential, δW, respectively. Specializing Eq. (4) to the case of small strains yields

$$\delta W = \int_{S^*} \{p^*\}^T\{\delta u\}\, dS^* = \int_V \{\sigma\}^T\{\delta\varepsilon\}\, dV \tag{5}$$

where the first integral is over the deformed surface area, S^*, and stresses $\{\sigma\}$ are now the usual stresses referred to the undeformed body. Equation (5) is restricted to small strains but large displacements are allowed.

The development of the equilibrium equations in finite element form is of course a straight forward procedure from this point. Following the notation of Ref. 6, the strain energy of the element is written using nonlinear strain-displacement relations and is conveniently separated into two parts

$$U = U_L + U_{NL} \tag{6}$$

where U_L is the strain energy based on linear strain-displacement relations and U_{NL} is the strain energy contribution due to the inclusion of nonlinearities in the strain-displacement relations. The equations of equilibrium are then obtained from Eq. (5) as

$$[K]\{q\} = \{Q\} - \{\partial U_{NL}/\partial q\} \tag{7}$$

where $[K]$ is the usual linear stiffness matrix obtained from U_L, $\{Q\}$ is a matrix of generalized forces (which may be functions of the displacements), and the last term of Eq. (7) is a matrix of pseudo generalized forces due to the nonlinearities. For simplicity, Eq. (7) is rewritten

$$[K]\{q\} = \{Q\} - \{Q^*(q)\} \tag{8}$$

where

$$\{Q^*(q)\} = \{\partial U_{NL}(q)/\partial q\} \tag{9}$$

Equation (8) represents a system of nonlinear algebraic equations which must be solved for any generalized load vector $\{Q\}$.

Although the equilibrium equations represented by Eq. (8) were developed in a finite element sense, a finite-difference formulation would lead to a similar system of nonlinear algebraic equations. Consequently, the solution procedures to be developed in the next section will apply equally well to both finite element and finite-difference formulations.

Solution Procedures

While the development of the basic equilibrium equations is a straightforward procedure, the solution of these equations is a more difficult task. Success in obtaining an accurate solution often depends primarily upon the solution procedure that one uses. In this section, the solution procedures discussed in a prior section and several new techniques will be cast into the notation of the previous section. Application and evaluation of the solution procedures will be presented in the next section.

Incremental stiffness procedure

In order to solve Eq. (8), it is assumed that a solution $\{q_o\}$ is known at load $\{Q_o\}$ and that a solution $\{q_o + \Delta q\}$ is desired at a load $\{Q_o + \Delta Q\}$. Substituting these values into Eq. (8) yields

$$[K](\{q_o\} + \{\Delta q\}) = \{Q_o\} + \{\Delta Q\} - \{Q^*(q_o + \Delta q)\} \tag{10}$$

Using a first-order Taylor series expansion on the term $\{Q^*\}$ yields

$$\{Q^*(q_o + \Delta q)\} = \{Q^*(q_o)\} + [\partial^2 U_{NL}(q_o)/\partial q_i \partial q_j]\{\Delta q\} \tag{11}$$

Substituting Eq. (11) into Eq. (10) and rearranging leads to

$$([K] + [K^*(q_o)])\{\Delta q\} = \{\Delta Q\} + \{Q_U(q_o)\} \tag{12}$$

where

$$[K^*(q)] = [\partial^2 U_{NL}(q)/\partial q_i \partial q_j] = [\partial Q_i^*(q)/\partial q_j] \tag{13}$$

and

$$\{Q_U(q)\} = -[K]\{q\} + \{Q\} - \{Q^*(q)\} \tag{14}$$

The last term of Eq. (12) represents the equilibrium equation and is zero for a true equilibrium position $\{q_o\}$. Consequently, Eq. (12) reduces to

$$([K] + [K^*(q_o)])\{\Delta q\} = \{\Delta Q\} \tag{15}$$

This equation allows a solution to be obtained at a point one load step beyond the load at which a solution is already known.

Self-correcting incremental procedure

The last term in Eq. (12) was discarded under the presumption that equilibrium was exactly satisfied at each stage of the incremenal loading. In general, equilibrium is not satisfied at each load step and consequently, the term $\{Q_U(q_o)\}$, defined by Eq. (14) represents an out-of-balance force which must be added to the structure to keep it in equilibrium. Consequently, the self-correcting incremental procedure becomes

$$([K] + [K^*(q_o)])\{\Delta q\} = \{\Delta Q\} + \{Q_U(q_o)\} \tag{16}$$

Newton-Raphson method

The generalized Newton-Raphson method may be applied to Eq. (8) with no difficulty. The equations are first rewritten as a set of functions for which the roots are desired

$$\{f(q)\} = [K]\{q\} - \{Q\} + \{Q^*\} = \{0\} \tag{17}$$

Based upon some initial estimate of the displacements $\{q^k\}$ at a given load $\{Q\}$, a first-order Taylor series expansion is used to determine $\{q^k + \Delta q^k\}$ such that $\{f(q^k + \Delta q^k)\}$ vanishes. This yields a system of equations for $\{\Delta q^k\}$

$$([K] + [K^*(q^k)])\{\Delta q^k\} = \{Q_U(q^k)\} \tag{18}$$

The correction $\{\Delta q^k\}$ is added to the approximate root $\{q^k\}$ to obtain a more nearly correct $(k+1)$th approximate solution

$$\{q^{k+1}\} = \{q^k\} + \{\Delta q^k\} \tag{19}$$

The new solution $\{q^{k+1}\}$ is substituted into Eq. (18) to obtain a further correction. The process is continued until $\{\Delta q^k\}$ is sufficiently small or until the unbalance in nodal forces $\{Q_U(q^k)\}$, is sufficiently small. For highly nonlinear problems, the usual procedure is to apply the load in increments and solve for the displacements at all intermediate load steps.

Modified Newton-Raphson procedure

Most investigators have found that the updating of the coefficient matrix $[K + K^*(q)]$ in Eq. (18) at every iteration can become time consuming even on the best of computers. Not only is the evaluation of $[K^*]$ lengthy, but each time $[K^*]$ is updated, the new coefficient matrix must be inverted. Consequently, some researchers[14–16] use a modified Newton-Raphson procedure wherein the $[K^*]$ matrix is held constant for several iterations or load increments and, after the rate of convergence has begun to deteriorate, the matrix is updated based on the current displacements.

In addition, various extrapolation procedures are often incorporated to predict more accurately the initial displacement estimates when going from one load increment to the next. Under-relaxation is often applied to the displacements to accelerate convergence. Various schemes to automatically reduce or increase the load step size may be incorporated into the computer code.[31]

This use of such procedures places some burden upon the user in deciding at what iteration the matrix should be updated, when the load step size should be changed, etc. If one is not careful, the procedure may diverge if the matrix is not updated often enough.

Perturbation method

The perturbation method has been investigated by several researchers.[12,13] The equilibrium equations can be written in the form

$$[K]\{q\} = \bar{P}\{\bar{Q}\} - \{Q^*(q)\} \tag{20}$$

where the displacements $\{q\}$ are now taken as functions of the load parameter, $\bar{P}$. At a known equilibrium point on the load displacement path, the derivatives of this path may be used to predict a further point. Expanding the displacements in a Taylor series about some known load-displacement state $(q, \bar{P})$ yields

$$q_i(\bar{P}+\Delta\bar{P}) = q_i(\bar{P}) + \dot{q}_i(\bar{P})\Delta\bar{P} + \tfrac{1}{2}\ddot{q}_i(\bar{P})\Delta\bar{P}^2 + \tfrac{1}{6}\dddot{q}_i(\bar{P})\Delta\bar{P}^3 + \ldots \tag{21}$$

where the dot denotes differentiation with respect to $\bar{P}$ and i refers to the degree of freedom. Differentiating Eq. (20) repeatedly and making use of the identity in Eq. (13) yields the set of linear equations

$$\begin{aligned} (K_{ij}+K_{ij}^*)\dot{q}_j &= \bar{Q}_i \\ (K_{ij}+K_{ij}^*)\ddot{q}_j &= -(\partial^2 Q_i^*/\partial q_j \partial q_k)\dot{q}_j\dot{q}_k \\ (K_{ij}+K_{ij}^*)\dddot{q}_j &= -3(\partial^2 Q_i^*/\partial q_j \partial q_k)\ddot{q}_j\dot{q}_k - \\ &\quad (\partial^3 Q_i^*/\partial q_j \partial q_k \partial q_1)\dot{q}_j\dot{q}_k\dot{q}_1 \end{aligned} \tag{22}$$

etc. All terms in Eqs. (22) are evaluated based on the known equilibrium point $(q, \bar{P})$. It should be noted that index summation notation has been used in writing Eq. (22). The sets of equations are solved sequentially for as many terms as are desired in the Taylor expansion. The series is then evaluated to obtain the displacements at the point $(\bar{P}+\Delta\bar{P})$. The next point on the load-displacement curve can be obtained by expanding about the point just determined.

This procedure, without any modifications, will tend to drift from the true curve since errors at each step will accumulate. This problem may be overcome by adding a corrective cycle at each load increment. Walker[12] uses Newton-Raphson iteration to improve the solution while Connor and Morin[13] utilized a further Taylor series expansion to correct the solution. The perturbation method is not applicable to highly nonlinear problems unless modifications such as those made in Refs. (12) or (13) are included.

Some comments concerning the usability of this procedure are in order. The method is well suited for studying stability problems and for tracing the post-buckled path. However, it appears that the method may become extremely time consuming for all but very simple problems.

If one considers the analysis of shells of revolution by the finite element method using a Fourier expansion in the circumferential direction, it turns out that the evaluation of the terms on the right side of Eq. (22) is extremely time consuming because of the coupling which exists between harmonics. The same would be true for any problem involving many degrees of freedom. Comparing the amount of calculations to the Newton-Raphson method, the perturbation method probably requires an order of magnitude more calculations for any given load step (depending upon the number of terms retained in the expansion).

Initial-value approach

The nonlinear equilibrium equations given by Eq. (8) may be transformed into first-order ordinary differential equations. If the load is assumed to be a linear function of a single load parameter, $\bar{P}$, then the equilibrium equations take the form of Eq. (20)

$$[K]\{q\} = \bar{P}\{\bar{Q}\} - \{Q^*\} \tag{23}$$

Differentiating Eq. (23) with respect to the scalar $\bar{P}$ yields

$$[K]\{\dot{q}\} = \{\bar{Q}\} - \{\dot{Q}^*\} \tag{24}$$

where a dot denotes differentiation with respect to $\bar{P}$.

Making use of chain rule differentiation, the last term in Eq. (24) may be rewritten

$$\{\dot{Q}^*\} = [\partial Q_i^*/\partial q_j]\{\dot{q}_j\} = [K^*]\{\dot{q}\} \tag{25}$$

Substituting Eq. (25) into Eq. (24) yields

$$([K]+[K^*])\{\dot{q}\} = \{\bar{Q}\} \tag{26}$$

Equation (24) may be differentiated once more to obtain a second derivative form

$$[K]\{\ddot{q}\} = -\{\ddot{Q}^*\} \tag{27}$$

Equations (24, 26, and 27) represent ordinary differential equations and may be integrated using any one of a large number of integration schemes. It should be noted that Eqs. (24) and (27) do not contain the $[K^*]$ matrix whereas Eq. (26) does. Since the evaluation of the $[K^*]$ matrix is usually somewhat time consuming, it would appear that these two forms might be the most desirable. However, Eqs. (24) and (27) contain derivatives of the nonlinear pseudo load, $\{Q^*\}$, which may cause stability problems because of the finite differencing of the derivatives. For example, some plasticity analyses have used a forward Euler formula for $\{\dot{q}\}$ and a backwards Euler formula for $\{\dot{Q}^*\}$

$$\begin{aligned} \{\dot{q}_i\} &= (1/\Delta\bar{P})(\{q_{i+1}\} - \{q_i\}) \\ \{\dot{Q}_i^*\} &= (1/\Delta\bar{P})(\{Q_i^*\} - \{Q_{i-1}^*\}) \end{aligned} \tag{28}$$

Substituting into Eq. (24), the recurrence relation for $\{q_{i+1}\}$ at load step $i+1$ is given by

$$\{q_{i+1}\} = \{q_i\} + [K]^{-1}(\Delta\bar{P}\{\bar{Q}\} - \{Q_i^*\} + \{Q_{i-1}^*\}) \tag{29}$$

It has been found that the preceding finite-difference form becomes unstable for highly nonlinear problems.

Several finite difference forms of these equations which are presented in Ref. 32 will be discussed in the next section on application and evaluation of the solution procedure.

Self-correcting initial-value formulation

The solution obtained by Eqs. (24, 26, and 27) tends to drift away from the correct solution. Stricklin, Haisler, and Von Riesemann[26] have proposed several self-correcting initial-value formulations which reduce this drifting. In their formulation, the unbalance in equilibrium forces, $\{Q_U\}$, given by Eq. (14), is multiplied by an amplification factor Z and added to the right hand side of Eqs. (24) and (26) to yield

$$[K](\{\dot{q}\} + Z\{q\}) = (1+Z\bar{P})\{\bar{Q}\} - Z(\{Q^*\} + (1/Z)\{\dot{Q}^*\}) \tag{30}$$

$$([K]+[K^*])\{\dot{q}\} + Z[K]\{q\} = (1+Z\bar{P})\{\bar{Q}\} - Z\{Q^*\} \tag{31}$$

Of course, the term that has been added to each equation, $\{Q_U\}$, is zero for a true equilibrium position, i.e., as long as the solution is on the true load-displacement curve. It should be pointed out that Eq. (31) with $Z = 1/\Delta\bar{P}$ is equivalent to the self-correcting incremental approach, Eq. (16). Noting that Eqs. (30) and (31) can be written as $\{\dot{Q}_U\} + Z\{Q_U\} = \{0\}$, it is seen that the unbalance, $\{Q_U\}$, is reduced exponentially to zero as $\bar{P}$ increases, and hence the terminology self-correcting is used.

Equation (27) is put into self-correcting form by adding the unbalance in force, $\{Q_U\}$, plus its first derivative with respect to $\bar{P}$, $\{\dot{Q}_U\}$, to the right-hand side to yield

$$[K]\{\ddot{q}\} = -\{\ddot{Q}^*\} + Z\{Q_U\} + C\{\dot{Q}_U\} \tag{32}$$

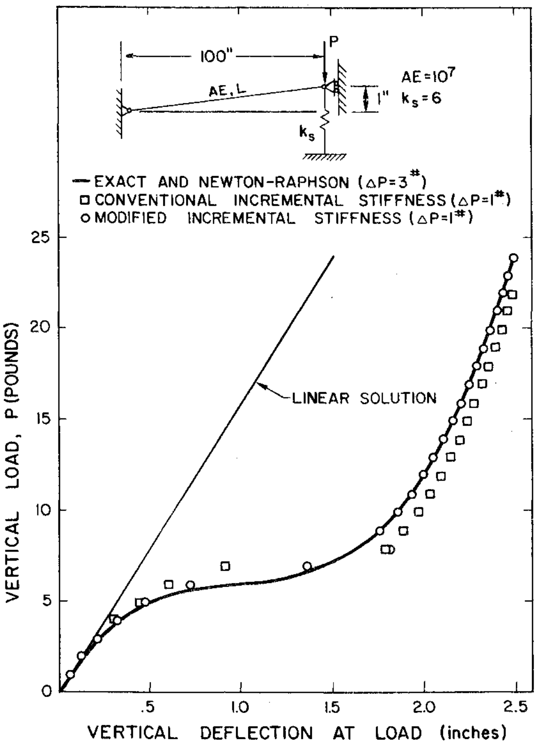

Fig. 1 Comparison of conventional and modified incremental stiffness solutions for truss-spring problem.

where again Z and C are scalar amplifying factors to control the effect of the added terms. Substituting Eq. (14) into Eq. (32) yields

$$[K](\{\ddot{q}\}+C\{\dot{q}\}+Z\{q\}) = (C+Z\bar{P})\{\bar{Q}\} - Z(\{Q^*\}+(C/Z)\{\dot{Q}^*\}+(1/Z)\{\ddot{Q}^*\}) \quad (33)$$

Equation (33) has a simple physical interpretation which aids in its solution. The left-hand side is equivalent to the equations for simple harmonic motion with a damping coefficient C and an undamped natural frequency of $(Z)^{1/2}$ rad/lb. The right-hand side is a forcing function. A more precise discussion of this formulation is presented in Ref. 27.

It should be noted that the use of Eq. (30) or (33) requires only a single inversion of the stiffness matrix for the entire solution. Other forms of these equations may be obtained by using Eq. (25), however, this would introduce the $[K^*]$ matrix and consequently, the computational efficiency of Eqs. (30) or (33) would be lost.

Evaluation of Solution Procedures

In order to compare and evaluate the solution procedures presented in the previous section, a number of highly nonlinear structural problems were considered. Each particular test case was solved using each solution technique in an attempt to determine which procedure is the best. In general, the procedure that was followed was to test the solution methods on very simple beam and truss problems and then apply the most promising techniques to more difficult problems. The procedures which appeared best were coded into an existing finite element shell of revolution program, SNASOR II.[31] Both symmetrically and asymmetrically loaded shells of revolution exhibiting highly nonlinear behavior were ultimately solved.

Truss-Spring Problem

Initially, the very simple one degree-of-freedom truss-spring problem shown in Fig. 1 was investigated. Although this problem is a rather elementary one, it was chosen because of the small amount of computational effort involved and because its solution will provide insight into the expected behavior of the solution procedures when applied to more difficult problems.

Figure 1 presents the load deflection curve for the truss-spring problem as obtained by the Newton-Raphson method, Eq. (18), the incremental method, Eq. (15), and the self-correcting incremental method, Eq. (16). Buckling and bending of the member have been suppressed and only elastic behavior is considered. As can be seen, the problem is highly nonlinear and exhibits large deflection behavior typical of that which one wishes to be able to analyze. As would be expected, the Newton-Raphson method solves this problem with no difficulty. Using load increments of one pound, both incremental procedures tend to drift away from the correct solution in the highly nonlinear portion of the curve. The self-correcting§ incremental stiffness procedure, Eq. (16), corrects itself significantly and returns to the correct solution after the flat portion of the curve is passed. For this case, then, the small additional computation time involved in applying the self-correcting form is well worth the effort.

It was attempted to solve this problem by simple iteration on the equilibrium equations, Eq. (8). Iterating at load increments of one-tenth pound, the procedure converged and agreed completely with the exact solution. However, a very large number of iterations was required at each load value to obtain convergence (thirty to forty) and the procedure refused to converge past a load of twenty-two pounds.

The treatment of the nonlinear problem as an initial-value problem opens the door to a large number of solution procedures that can be considered. Figure 2 presents solutions obtained by integrating Eq. (26) with the fourth-order Runge-Kutta and Adams predictor-corrector methods.[34,35] The fourth-order Runge-Kutta solution agrees quite well with the exact results using a load increment of one pound. Lower-order Runge-Kutta methods tended to drift away from the correct solution for the one pound load increment. Various predictor-corrector methods were investigated. The fourth-order Milne predictor-corrector method required a large number of iterations to converge and was very unstable for the one pound load increment. It

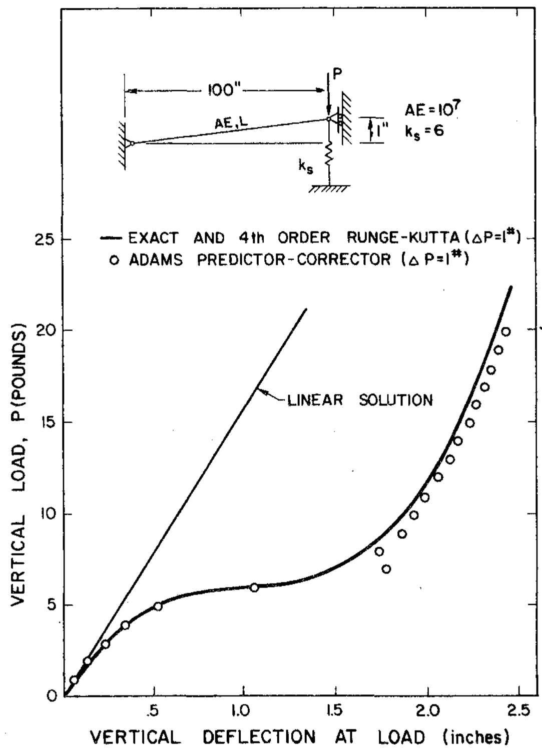

Fig. 2 Comparison of initial-value solutions for truss-spring problem.

§ On this and several figures to follow the self-correcting incremental procedure is sometimes denoted as the modified incremental procedure.

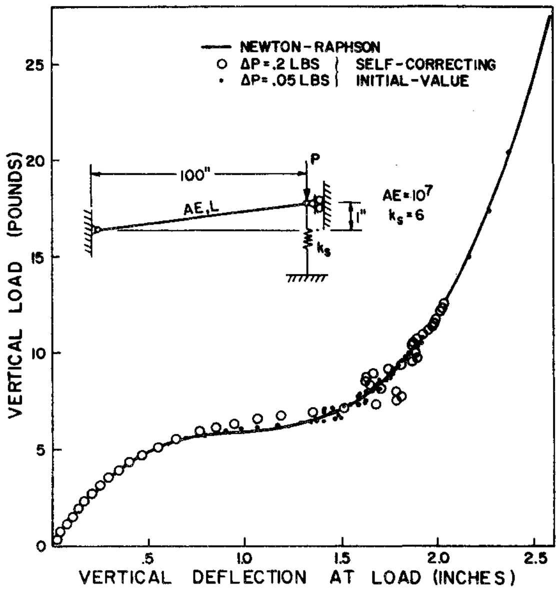

Fig. 3 Self-correctiong initial-value solution for truss.

tended to converge to nonequilibrium solutions such that the converged solutions oscillated about the true curve. Of course, most text books point out that the Milne method is inherently unstable. The oscillations still remained for a load increment of one-fourth pound. Reducing the error criterion did not seem to negate the oscillations. The Hamming predictor-corrector solution exhibited this same behavior but only at the load range of six to eight pounds where the curve is very flat. The Adams fourth-order predictor-corrector appeared to be the best of the predictor-corrector approaches that were tried. For the one pound load increment, it converged to nonequilibrium positions as seen in Fig. 3, but did not exhibit the oscillation of the Milne method. For a load increment of one-fourth pound, the Adams procedure gave results identical to the Newton-Raphson solution and remained stable to two-hundred pounds where the solution was stopped.

Of the two first derivative initial-value formulations, Eq. (26) is the most accurate; however, Eq. (24) is appealing since it does not involve the evaluation of $[K^*]$. Many finite difference forms were used to approximate the integration of Eq. (24). A trial and error procedure was followed in trying to find the best way to solve Eq. (24). Various finite difference forms of $\{\dot{q}\}$ were tried with various finite difference approximations of $\{\dot{Q}^*\}$. The $\{\dot{Q}^*\}$ term was also approximated by linear extrapolation in some cases. One procedure that appeared to be quite accurate was obtained by using a three point backward formula for $\{\dot{q}\}$

$$\{\dot{q}_i\} = (1/2\Delta\bar{P})(3\{q_i\} - 4\{q_{i-1}\} + \{q_{i-2}\}) \tag{34}$$

The term $\{\dot{Q}^*\}$ is first evaluated by linear extrapolation from previous load increments

$$\{\dot{Q}_i^*\} = 2\{\dot{Q}_{i-1}^*\} - \{\dot{Q}_{i-2}\} \tag{35}$$

and then the derivatives are approximated using the same formula as for the displacements. This yields the recurrence relation

$$\{q_{i+1}\} = \tfrac{1}{3}(4\{q_i\} - \{q_{i-1}\}) + [K]^{-1}(\tfrac{2}{3}\Delta\bar{P}\{\bar{Q}\} - 2\{Q_i^*\} + \tfrac{11}{3}\{Q_{i-1}^*\} - 2\{Q_{i-2}^*\} + \tfrac{1}{3}\{Q_{i-3}^*\}) \tag{36}$$

For a load increment of one-tenth pound, almost exact agreement was obtained with the Newton-Raphson solution. However, numerical stability problems were encountered when the load approached twenty-three pounds. In general, extrapolation of the pseudo loads $\{\dot{Q}^*\}$, is not desirable as it usually causes numerical instability.

While it was found that several finite-difference forms were quite accurate, no combinations of finite difference approximations could be found so that the solution of Eq. (24) remained absolutely stable in the highly nonlinear range (for example, when 100–200 lb are applied in this case).

The self-correcting initial-value formulations were studied in some detail for the truss-spring problem. The first derivative initial-value formulation described by Eq. (31) was integrated using a fourth-order Adams predictor-corrector scheme. A load increment of one-half pound was used and $Z\Delta P$ was taken as one. The solution obtained corresponds almost exactly with the correct solution. In order to save computation time, the $[K^*]$ matrix was updated every ten iterations which in most cases meant that updating occurred every three or four load increments. The load increment of one-half pound appeared to be optimum since taking a larger load increment required a greater number of total iterations for a complete solution. Higher values of Z were tried but these tended to make the procedure unstable. In general, the self-correcting initial-value formulation, Eq. (31), was found to be better than the incremental stiffness method.

The study of Eq. (30) was somewhat frustrating in that no stable integration procedures could be found. Various finite difference forms were used for $\{\ddot{q}\}$ while the nonlinear pseudo forces, $\{Q^*\}$, were approximately by linear extrapolation from the previous load increments. Regardless of the form of $\{\ddot{q}\}$, the procedure gave a non-converging oscillating solution past about thirty-five pounds. In general, the procedure tended to always lead the correct solution. The oscillation is believed to be due to the omission of the mass-like term [which is included in Eq. (33)] which would tend to make the structure accelerate.

Equation (33) was solved by using a four point backwards difference formula (Houbolt) for $\{\ddot{q}\}$ and a three point backwards difference formula for $\{\dot{q}\}$. The $\{\dot{Q}^*\}/Z$ term was neglected on the assumption that it is small and the remaining nonlinear pseudo force terms were approximated by linear extrapolation from previous load increments.[26] A recurrence relation for $\{q_i\}$ at load step i (corresponding to load parameter $\bar{P}_i$) is obtained in Ref. 26 as

$$\{q_i\} = \left(\Delta\bar{P}^2[K]^{-1}\left\{(C+Z\bar{P}_i)\{\bar{Q}\} - \left[Q_{i-1}^* + \frac{Q_{i-1}^* - Q_{i-2}^*}{\Delta\bar{P}}\left(\Delta\bar{P} + \frac{C}{Z}\right)\right]Z\right\} + (5+2C\Delta\bar{P})\{q_{i-1}\} - \left(4+\frac{C\Delta\bar{P}}{2}\right)\{q_{i-2}\} + \{q_{i-3}\}\right) \Big/ \left[2 + \frac{3C\Delta\bar{P}}{2} + Z\Delta\bar{P}^2\right] \tag{37}$$

Starting values for the solution procedure were obtained by iteration on the equilibrium equations. After some experimentation, it was found that the parameters Z and C should for stability reasons be taken as decreasing functions of the load

$$Z = 10/\Delta\bar{P}[(\Delta\bar{P})\bar{P}]^{1/2}$$
and
$$C = Z^{\cdot 2}/2 \tag{38}$$

The particular form chosen here yields a Z which is quite large for low values of the load and becomes smaller as the value of the load is increased (and as higher nonlinearities are encountered). The equation chosen for C yields a value of C such that C/Z is small and always less than one. The factor C/Z represents the fraction of a load increment, $\Delta\bar{P}$, that the $\{Q^*\}$ generalized forces are extrapolated and consequently this term should remain small.

Figure 3 presents the solution obtained with Eq. (37) for load increments of 0.2 and 0.05 lb. In both cases, the oscillations in the solution quickly damp out and the solution returns to the correct curve.

Shallow Cap—Symmetrically Loaded

The second problem considered is a shallow spherical cap ($\lambda = 6$) under a point load at the apex. Figure 4 indicates the behavior is highly nonlinear with an initial softening region and subsequent stiffening as the shell folds through. Results were obtained using the SNASOR-II computer code.[31]

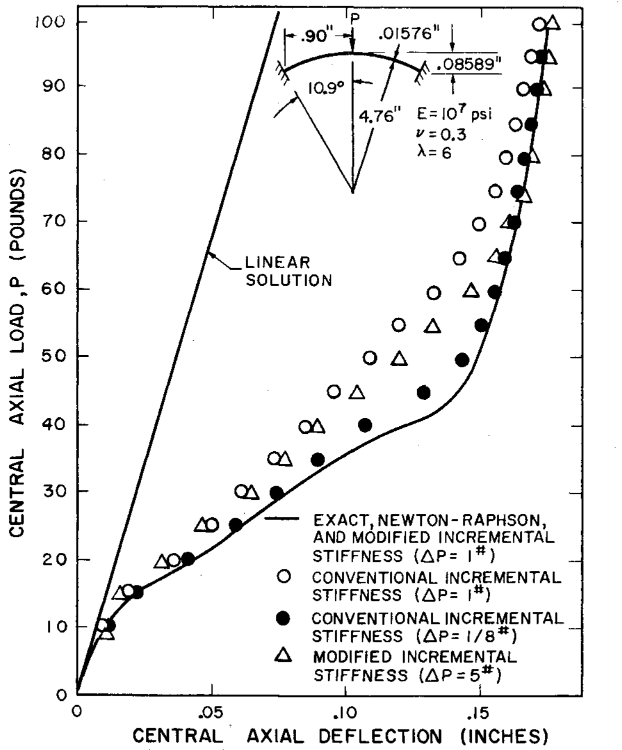

Fig. 4 Comparison of solution procedures for shallow spherical shell ($\lambda = 6$) with axial point load at apex.

The Newton-Raphson procedure, Eq. (18), yields results which agree with finite-difference results published by Mescall[36] for load increments of one pound and five pounds. The results of the conventional incremental stiffness procedure, Eq. (15), for load increments of one pound and one-eighth pound are also shown in Fig. 4. It is obvious that unless a sufficiently small load increment is used, the solution will drift away from the true solution. The conventional incremental stiffness procedure for a load increment of one-eighth pound required more than ten times the amount of computer time required by the Newton-Raphson procedure using a five pound load increment. The modified self-correcting incremental procedure, Eq. (16), tends to exhibit much better convergence characteristics in comparison to the conventional form. For a load increment of one pound, agreement is obtained with the exact solution. For a load increment of five pounds the self-correcting (modified) form drifts somewhat. It should be noted that direct iteration on the equilibrium equations failed to converge past twenty pounds and with an under-relaxation scheme would converge only up to thirty pounds.

The initial-value formulation of Eq. (24) was integrated using an Adams second-order predictor-corrector procedure. Although the procedure yielded very accurate results with a load increment of one-fourth pound, it refused to converge past a load of twelve pounds. In all likelihood, one could probably apply the

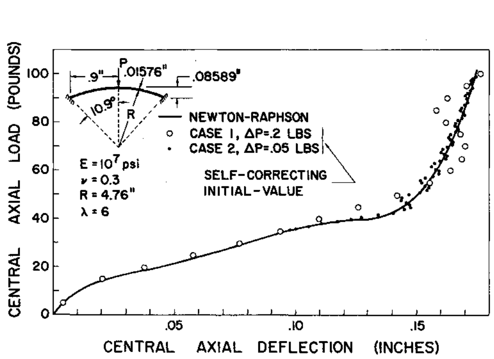

Fig. 5 Self-correcting initial-value solutions for shallow cap.

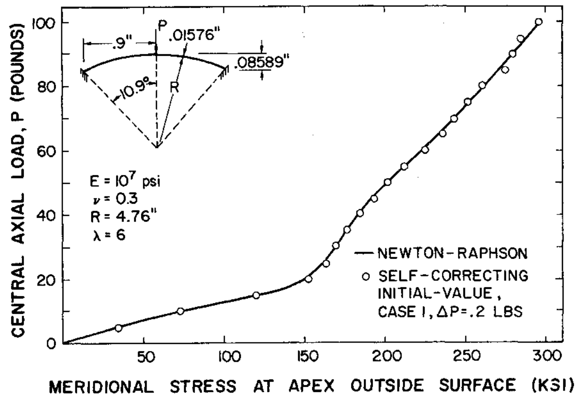

Fig. 6 Meridional stress for shallow cap using self-correcting initial-value method.

second-order Adams procedure to Eq. (26) which uses the $[K^*]$ matrix and integrate to a higher load. However, this was not tried since the evaluation of $[K^*]$ considerably adds to the computation time. Instead, more attention was focused toward procedures not using the nonlinear stiffness matrix.

The self-correcting initial-value formulation, Eq. (37), provided some very interesting results for the shallow cap as seen in Fig. 5. Case 1 corresponds to $Z = 0.6/\{\Delta P[P(\Delta P)]^{1/2}\}$ and $C = 0.2(\Delta P)Z$ and Case 2 uses $Z = 5/[\Delta P(P\Delta P)^{1/2}]$ and $C = Z^{\cdot 2}/2$. The results are remarkably accurate considering the high degree of nonlinearity exhibited. In both cases, the solution oscillates about the true solution with the oscillations subsiding after a few cycles.

Stresses were also calculated for this problem. They tended to oscillate about the true solution but the oscillations were much less pronounced as seen in Fig. 6. In some cases, the stress oscillations were slightly out of phase with the displacement oscillations.

Shallow Cap—Asymmetric Loading

A third example considers the same shallow cap ($\lambda = 6$) but with a vertical point load applied off-center. The problem is extremely nonlinear with the nonlinear theory predicting an axial displacement at the apex which is 20 times that predicted by linear theory. Only three Fourier harmonics were used in the analysis which does not of course yield an exact solution to the point load case considered here. The Newton-Raphson solution is shown in Fig. 7 along with three self-correcting initial-value solutions [Eq. (37)]. Equation (37) was solved using the parameters shown in Table 1 where a $\Delta\bar{P}$ of 0.1 corresponds to a load increment of 0.1 lb. It is interesting to note that the Newton-Raphson solution required approximately 16 min of IBM 360/65 computer time while the self-correcting initial-value

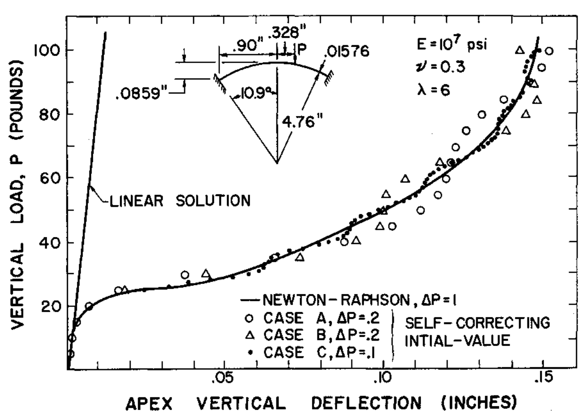

Fig. 7 Self-correcting initial-value solution for asymmetrically loaded shell.

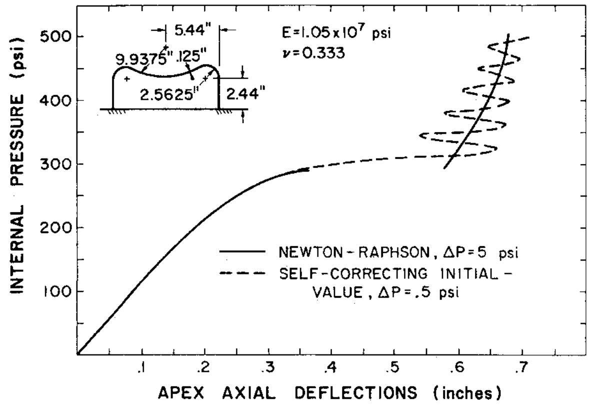

Fig. 8 Buckling analysis of a negative Gaussian curvature shell.

formulation, Case B, required 6 min and Case C required less than 12 min. Considering the accuracy of Case C, one would have to conclude that the initial-value formulation is certainly worthy of consideration as a practical solution technique.

Snap Buckling Problem

The last case considers the snap buckling of the negative Gaussian curvature shell shown in Fig. 8. The Newton-Raphson procedure with a load increment of five pounds was used to obtain the solution. This is compared to the solution obtained with the self-correcting initial-value formulation of Eq. (37) using the same Z and C parameters as Case C of Table 1. With respect to the initial-value formulation, it should be noted that the total load was scaled such that the load parameter, $\bar{P}$, ranged from 0 to 100. This allowed some uniformity to the solution procedure such that once values of $\Delta\bar{P}$, Z, and C were found that gave realistic results, these values could, in general, be used for most problems encountered regardless of the nonlinearity of the problem. Consequently, using a $\Delta\bar{P}$ of 0.1 with a total load of 500 psi gives the actual load increment as 0.5 psi.

The Newton-Raphson procedure has no difficulty in predicting the post-buckled behavior. In traversing the unstable region, the procedure converged at 290 psi and then jumped to the post-buckling position at 295 psi.

Likewise, the initial-value formulation, Eq. (37), yielded a very accurate solution in the prebuckled state and then went into the postbuckled shape at about 300 psi. Although the solution is oscillating somewhat, it is oscillating about the true curve and the oscillations appear to be damping out.

Although it was not done here, the oscillations could be damped out much faster by restarting the program at around 325 psi with a smaller load increment. As soon as the oscillations diminished, the load increment could then be increased.

Concluding Remarks and Recommendations

The over-all purpose of this paper has been to present the methods of solution available for the geometrically nonlinear structural problem and to evaluate the procedures for highly nonlinear problems. To conclude this discussion of solution procedures, distinguishing features of each formulation will be discussed and recommendations as to which procedures are best as based on the results of the current study will be made. The conclusions are given as if the procedures were incorporated into large-scale computer programs capable of handling a large number of degrees-of-freedom and highly nonlinear behavior. Some conclusions are of course extrapolated to the large-scale problem based upon experience gained primarily with the shell of revolution analyses.

1) For the geometrically nonlinear problem, it appears that the conventional incremental stiffness procedure is too prone to drifting from the true solution to be of any real value if one is interested in accurate results. Accuracy can be achieved only with very fine load step refinement and in most cases, requires more computer time than say the Newton-Raphson method but never achieves the same accuracy.

2) On the other hand, if one is interested in only a first estimate of the solution, than the incremental stiffness method provides the cheapest solution. The inexperienced user will generally favor the incremental method since it requires only the specification of a load step size. However, when using this method, the user must be aware of possible drifting and must exercise caution regarding suitable step size.

3) The addition of the corrective term to the incremental procedure has been found in all cases to increase accuracy so as to be well worth the slight additional computational effort. This self-correcting incremental procedure enjoys the ease of usage of the conventional incremental procedure and approaches the accuracy of the Newton-Raphson method.

4) Procedures utilizing iteration or successive substitution to solve the equilibrium equations directly appear to be somewhat limited in applicability. The authors' experience with pure iteration is that it is applicable only when the nonlinear displacement solution differs from the linear value by less than a factor of two and one-half. At greater nonlinearities, the procedure will simply not converge. This procedure is extremely fast and is probably best used to obtain starting values for other solution techniques.

5) The Newton-Raphson method is probably the most accurate method available to date. Unfortunately, it is also the most expensive from the standpoint of computer expenditure because of the constant updating and inverting of the coefficient matrix. Although various modified Newton-Raphson procedures reduce the solution cost somewhat, they also require somewhat of a more experienced computer program user. These programs often require such additional input as how often to update the coefficient matrix, when to reduce the load step size, how and when to apply extrapolation and relaxation, etc. which all tends to require that the program user be thoroughly familiar with nonlinear analysis. However, if these parameters are established, then the modified Newton-Raphson method can be utilized by less experienced users to get accurate results regardless of step size selected.

6) Although the perturbation technique has not been specifically applied to any problems by the authors, it would appear that the procedure would be extremely time consuming when applied to a problem with substantial degrees of freedom and coupling between degrees of freedom.

7) The new initial-value formulations presented herein and elsewhere by the authors appear to be worthy of further investigation. If one is willing to accept the slight oscillations about the true solution, than the self-correcting forms provide results which are as accurate as the Newton-Raphson method and, in almost all cases considered, require less computer time than the Newton-Raphson method. Their computational efficiency arises from the fact that the stiffness matrix is inverted only once for an entire analysis. The initial-value procedures are quite easy to apply requiring only the application of the step size plus the parameters Z and C (although with the load scaled so that $\bar{P}$ ranges from 0–100, these parameters do not change much regardless of the nonlinearity). The amplitude of the oscillations can be reduced by reducing the step size. Recently, the authors have reported a procedure in Ref. 27 which completely eliminates these oscillations.

Table 1 Solution parameters for self-correcting initial-value solution, Eq. (37)

Case	$\Delta\bar{P}$	$Z(\Delta\bar{P})[\bar{P}(\Delta\bar{P})]^{1/2}$	C
A	0.2	1	$0.2(\Delta P)Z$
B	0.2	2	$0.2(\Delta P)Z$
C	0.1	5	$0.05(Z)^{1/2}$

8) In summary, if one is considering problems of only slight nonlinearity, then either the self-correcting incremental procedure or pure iteration will provide accurate solutions in reasonable computer times. For highly nonlinear problems, either the Newton-Raphson or self-correcting initial-value procedure is applicable. Which of the two that one chooses would appear to be determined by two factors: user familiarity with the computer code and amount of computer dollars available. For the user with ample computer funds, the Newton-Raphson method would be the best choice since it almost always gives very reliable solutions. However, for the inexperienced user, the self-correcting initial-value formulation appears to be a good alternative since it requires very little user know-how and is economical. Going one step further and considering general purpose computer codes where a large number of degrees of freedom are often used, the self-correcting initial-value formulation would be a good choice. It provides reasonably accurate results but requires much less time than the Newton-Raphson method. This is of course true because the stiffness matrix is inverted only once in the entire analysis whereas in the Newton-Raphson method the stiffness matrix must be formed and inverted many times.

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